1 Introduction

Physics is the study of the natural world through the creation and manipulation of models. Classical mechanics, for example, models the universe as a large, though finite, number of particles interacting through forces in predictable ways. Since the advent of quantum mechanical theory and general relativity, we have known that this model has its flaws, but it is still a useful way of describing and explaining our observations of the world. Even today, almost every student of physics begins with classical mechanics and builds more accurate models upon that foundation.

Likewise, many students may find visual analogies useful aids to intuition and understanding in more abstract areas of physical theory, especially when dealing with electricity and magnetism. For the sake of this paper, I will focus on the analogy of a ball moving on a spinning saddle as a mechanical representation of an ion in an RF-electric-quadrupole trap.

1.1 Ion Traps

If one desires to study the most basic building blocks of matter, it is useful to be able to keep one of said building blocks relatively still for an extended period of time. Ion traps allow just this sort of confinement: a charged atom can literally be trapped within a small radius, allowing more accurate study of its properties. In 1989, Wolfgang Paul received a share of the Nobel Prize in Physics for his development of the RF-electric-quadrupole ion trap, also called the Paul-type trap in his honor. This type of trap is especially useful in determining atomic masses—the trap itself can be used as a mass spectrometer [2].

Ion traps are also used to confine atoms for the purpose of high-accuracy frequency standards in atomic clocks [1], and in the future, trapped ions may serve as optical qubits for quantum computation [3].

1.2 Derivation of Ion Trap Potential

In order to confine an ion to an area, it is necessary to subject the ion to a restoring force that will draw it back to the system’s origin. The simplest example of such a force is analogous to the spring force and takes the form

\[ \mathbf{F} = -c \mathbf{r} \] (1)

where \( \mathbf{F} \) is the force acting on the ion, \( \mathbf{r} \) is the position vector of the ion, and \( c \) is an arbitrary constant.

Following the derivation of Thompson, Harmon, and Ball [4], I will recall that the force on a charged particle is proportional to the electric field and integrate (1) to find

\[ \phi(x, y, z) = \frac{\phi_0}{2r_0^2}(\alpha x^2 + \beta y^2 + \gamma z^2) \] (2)

where \( \phi \) is the electric potential at a given point in the Cartesian plane, \( \phi_0 \) is a constant defining the magnitude of the potential, \( r_0 \) is the radius of confinement, and \( \alpha, \beta, \) and \( \gamma \) are constants that serve to determine the shape of the potential. In order to ensure the negative sign in (1), it is necessary for \( \alpha, \beta, \) and \( \gamma \) to be positive.

Additionally, the application of Gauss’ Law to free space requires that the integral of the electric field over a sphere must be zero. Thus,

\[ \alpha + \beta + \gamma = 0 \] (3)
Figure 1: "Flapping Saddle" potential [4]

It appears impossible to satisfy (3) while still ensuring a trapping potential. To quote The Princess Bride, "But if there can be no arrangement, then we are at an impasse" [5]. There is, however, still the possibility of an arrangement, as I have not yet attempted to vary the potential with time. After setting \( \alpha = -\beta = 0 \) and \( \gamma = 0 \), and allowing \( \phi_0 \) to vary sinusoidally with time, (2) becomes

\[
\phi(x, y, z, t) = \frac{U_{RF} \cos(\Omega t)}{r_0^2} (x^2 - y^2)
\]

where \( U_{RF} \) is the amplitude of the ac component of the electric field applied to create the trap, and \( \Omega \) is the frequency of variation.

This field can be visualized as a "flapping saddle"–a quadrupole potential in which extrema oscillate between being peaks and being valleys. In order to predict the motion of the ion, Newton’s laws can be applied to (4) to produce the following differential equations:

\[
m \frac{d^2 x}{dt^2} = -\frac{2eU_{RF}}{r_0^2} \cos(\Omega t) x \\
\frac{d^2 y}{dt^2} = -\frac{2eU_{RF}}{r_0^2} \cos(\Omega t) y
\]

where \( m \) is the mass of the ion and \( e \) is the electron charge. These equations are of the form of the Mathieu differential equations and admit stable solutions whenever [4]

\[
\left| \frac{4eU_{RF}}{mr_0^2\Omega^2} \right| < 0.908
\]

It is interesting to notice that this stability condition is independent of the initial position and velocity of the trapped ion, depending only on the shape of the trap and the frequency of oscillation.

2 The Spinning Saddle Potential

Although not a perfect analogue to the RF-electric-quadrupole ion trap, the spinning saddle has the advantage of being visible, tangible, and easy to manipulate without the need for complicated equipment. In place of the ion moving within an electric quadrupole potential, I will now consider the problem of a ball moving within a gravitational quadrupole potential of the form

\[
U(x, y) = \frac{mgh_0}{r_0^2} (x^2 - y^2)
\]

where \( U \) is the gravitational potential, \( m \) is the mass of the particle, \( g \) is the acceleration due to gravity, \( r_0 \) is the radius of the saddle, and \( h_0 \) is the maximum height at that radius, measured from the inflection point of the saddle.

It is difficult to create a flapping saddle on which the ball can move, but the time-dependent potential of a saddle spinning at angular frequency \( \Omega \) is similar enough to the flapping potential as to be useful [4]:

\[
U(x, y, t) = \frac{mgh_0}{r_0^2} \left( (x^2 - y^2) \cos(2\Omega t) + 2xy \sin(2\Omega t) \right)
\]
Again following the derivation of Thompson, Harmon, and Ball [4], I apply Newton’s laws to get the coupled differential equations

\[ \frac{d^2}{d\tau^2} x + 2\bar{q}[x \cos(2\tau) + y \sin(2\tau)] = 0 \quad (10) \]

and

\[ \frac{d^2}{d\tau^2} y + 2\bar{q}[y \cos(2\tau) + x \sin(2\tau)] = 0 \quad (11) \]

where \( \tau = \Omega t \) and \( \bar{q} = \frac{gh_0}{r_0^2\Omega^2} \). These equations differ from the ion trap equations only by the coupled sin term and their solutions are similar in that they predict stable motion for \( \bar{q} \leq \frac{1}{2} \) regardless of the initial position and velocity of the ball.

2.1 Theoretical Trajectories

For a saddle with dimensions \( r_0 = 7.5 \) cm and \( h_0 = 1.25 \) cm, the critical frequency of rotation is \( f = 1.05 \) hz (this corresponds to angular frequency \( \Omega = 6.6 \) radians per second). I used Mathematica 10.1 [6] to find numerical solutions to and graph theoretical trajectories given by (10) and (11) for various rotational frequencies and initial conditions. A few examples appear as Figures 2 and 3.

![Figure 2: Theoretical trajectories from the initial position (1 cm, 1 cm) with zero initial velocity](image)

(a) \( f = 0.735 \) hz  (b) \( f = 1.05 \) hz (critical frequency)  (c) \( f = 1.365 \) hz

Figure 2: Theoretical trajectories from the initial position (1 cm, 1 cm) with zero initial velocity

![Figure 3: Theoretical trajectories at the critical frequency with zero initial velocity](image)

(a) initial position (1 cm, 0.25 cm)  (b) initial position (1 cm, 0.5 cm)  (c) initial position (1 cm, 0.5 cm)

Figure 3: Theoretical trajectories at the critical frequency with zero initial velocity
2.2 The Effect of Friction

In the course of their attempts to confine a ball to a spinning saddle, Thompson, Harmon, and Bell [4] noticed that, although the ball appeared to demonstrate stable behavior, its lifetime of confinement to the spinning saddle was nonetheless finite. Justified by their observation that balls coated in Teflon to reduce the coefficient of friction remained trapped for significantly longer lifetimes, Thompson, Harmon, and Bell incorporated the effect of friction into their model.

In the laboratory frame, the force of friction on the ball is

\[ F_x = -\alpha \left( \frac{d}{dt} x(t) + \Omega y(t) \right) \]  
\[ F_y = -\alpha \left( \frac{d}{dt} y(t) - \Omega x(t) \right) \]

where \( F_x \) and \( F_y \) are the components of the frictional force in the \( x \) and \( y \) directions, respectively, and \( \alpha \) is the coefficient for velocity-dependent sliding friction, measured in \( \text{kg s}^{-1} \). Thus the equations of motion become

\[ \frac{d^2}{dt^2} x = -2q[x \cos(2\tau) + y \sin(2\tau)] + \frac{\alpha m}{\Omega^2} \left( \frac{d}{dt} x(\tau) + y(\tau) \right) \]  
\[ \frac{d^2}{dt^2} y = -2q[y \cos(2\tau) + x \sin(2\tau)] + \frac{\alpha m}{\Omega^2} \left( \frac{d}{dt} y(\tau) - x(\tau) \right) \]

where, again, \( \tau = \Omega t \).

The addition of the friction term destabilizes the solutions to the differential equation, providing a theoretical explanation for the finite lifetime of the ball in the trap. The degree of the instability increases with \( \frac{\alpha m}{\Omega^2} \). I used Mathematica 10.1 [6] to find numerical solutions to these new differential equations and graph theoretical trajectories for various values of this ratio. Some examples appear as Figure 4.

![Figure 4: Theoretical trajectories at the critical frequency with zero initial velocity and initial position (1 cm, 1 cm)](image)

When friction is a factor, the lifetime of the ball within the trap is dependent on initial position and velocity, as well as mass, coefficient of sliding friction, and frequency.

3 Experiment

For the sake of this experiment, a hyperbolic paraboloid with dimensions \( r_0 = 7.5 \text{ cm} \) and \( h_0 = 1.25 \text{ cm} \) was modeled by using Mathematica 10.1 [6] to plot the intersection of \( z \leq \frac{1}{15}(x^2 - y^2) \) and \( x^2 + y^2 \leq 7.5 \). This graphic was then exported and 3D printed using a CubePro Duo two-headed printer. Despite precautions, one edge of the saddle pulled up during the course of the printing, creating an uneven bottom surface. This problem was remedied by propping the shorter side using duct tape until the saddle sat levely.
The saddle was thoroughly sanded to minimize friction, and then mounted on a circular platform connected to an adjustable-speed Unite MY6812 motor and a system of gears in the form of a Bodine Electric NCI-12R gear motor to decrease frequency and transform horizontal rotation to vertical rotation. A photograph of the system appears as Figure 5.

![Experimental set-up](image)

Figure 5: Experimental set-up

### 3.1 Trapping lifetimes

As expected based on the work of Thompson, Harmon, and Ball [4], the experimental confinement of a ball to the spinning saddle lasted for a finite length of time. In order to collect lifetime data, I first measured the frequency of rotation by documenting the amount of time the saddle took to complete ten rotations (a flap of tape was attached to the side of the platform to simplify this measurement). A ball was then placed as close to the center of the saddle as possible, and the duration of its confinement within the saddle’s radius was timed. Figure 6 displays the lifetimes for various ball types when the saddle was rotating at 1.4 hz, well above the theoretical critical frequency. The diameter and mass of each ball type appear in Table 1. There was no clear correlation between either of these parameters and the trapping lifetime—as Thompson, Harmon, and Ball [4] might predict, the lifetime is probably more dependent on the frictional coefficient between the ball and the saddle’s surface, a quantity which I have not developed a simple way to measure.

![Comparison of lifetimes for different ball types](image)

Figure 6: Comparison of lifetimes for different ball types. Five trials were taken for each ball. The average of these trials is displayed in orange.
Table 1: Diameter and mass for ball types

<table>
<thead>
<tr>
<th>Ball type</th>
<th>Mass (g)</th>
<th>Diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ping pong</td>
<td>1.8</td>
<td>39.6</td>
</tr>
<tr>
<td>large ball bearing</td>
<td>31.0</td>
<td>18.9</td>
</tr>
<tr>
<td>small ball bearing</td>
<td>3.6</td>
<td>9.6</td>
</tr>
<tr>
<td>billiard</td>
<td>150.3</td>
<td>54.0</td>
</tr>
<tr>
<td>plastic</td>
<td>75.9</td>
<td>61.9</td>
</tr>
<tr>
<td>tennis</td>
<td>60.0</td>
<td>62.5</td>
</tr>
</tbody>
</table>

Due to the sensitivity of the trajectory to initial position and velocity, the range of lifetimes for each type of ball is very large, but in this and subsequent experiments, the ping pong ball and the billiard ball both perform well, while the tennis ball consistently remains confined for only very short lifetimes. This observation suggests that friction may indeed be the main cause of the destabilization; both the ping pong ball and the billiard ball appear to have a very low coefficient of friction, while the tennis ball has a high coefficient of friction and hardly slips at all.

I also collected lifetime data for each type of ball at a variety of frequencies. The results of this experiment appear as Figure 7.

These plots show good evidence for a critical frequency above which the trapping lifetime increases dramatically. This frequency, however, appears to be closer to 1.3 Hz than to the theoretical 1.05 Hz predicted by (12) for most ball types. A possible explanation for this deviation may be the non-infinitesimal radius of the balls—the model I have been using treats the ball as a point particle within the saddle. Attempting to incorporate the radius of the ball into the equations of motion is prohibitively complex, but it is worth noting that the deviation of the experimental critical frequency from the theoretical value appears at least partially dependent on the radius of the ball. See Figure 8.

3.2 Experimental Trajectories

In order to plot experimental trajectories, I mounted a Casio EX-FH25 high speed digital camera approximately 1 meter above my spinning saddle set-up. Video was taken at 240 frames per second and analyzed using Physmo software [7]. This analysis consisted of tracking the position of the ball with respect to the center of the saddle. Due to time constraints, the position of the ball was only marked...
about every 7 frames, corresponding to about one measurement every 0.3 seconds. The position of the flap was also marked in order to achieve a more accurate frequency measurement for this data.

In order to compare experimental to theoretical trajectories, it is necessary to know initial x- and y-velocities. Initially, these were calculated by taking the average velocity over the first 0.2 seconds, but theoretical trajectories matched the experimental trajectories more closely if it was assumed that $v_{xi} = v_{yi} = 0$. Some examples of experimental trajectories compared to theoretical trajectories appear as Figures 9 and 10. The coefficient of friction was estimated based on the length of confinement within a 7.5 cm radius of the origin.

Figure 9: Trajectories for the small ball bearing on a saddle rotating at 1.605 hz. The x- and y-axis are position measured in meters. Theoretical trajectories are plotted for 5 seconds, while the experimental trapping lifetime of the ball was 3.4 seconds.

Figure 10: Trajectories for the billiard ball on a saddle rotating at 1.622 hz. The x- and y-axis are position measured in meters. Theoretical trajectories are plotted for 10 seconds, while the experimental trapping lifetime of the ball was 9.9 seconds.

In general, there are two main differences between the experimental and theoretical trajectories. First,
the experimental trajectories display significantly less micromotion than the theoretical trajectories. This deviation may be partially due to a lack of accuracy in video analysis, but, similarly to the deviation from theoretical critical frequency, may also be due to the radius of the ball. Secondly, the balls appear to revolve far more quickly about the origin than the model predicts. The billiard ball, for example, completes 7 revolutions counterclockwise during the course of its 9.9 second confinement to the saddle, whereas the theoretical trajectory predicts 1.5 clockwise revolutions. Given the fact that the extra revolution is in the same direction as the rotation of the saddle, this excessive revolution suggests that the ball is clinging to the saddle more than the model, which includes only velocity-dependent sliding friction, accounts for.

4 Looking Forward

Based on the results of my experiment, it is clear that Thompson, Harmon, and Bell’s friction model [4] is inadequate to describe the motion of a ball on a spinning saddle. In order to completely understand this mechanical system, future researchers may wish to incorporate static friction into the theoretical model or use experimental data to develop a model that accounts for the radius of the ball. An experiment testing the trapping lifetimes of balls of the same mass and material but with differing radii would be particularly interesting. An interested student may also wish to examine how deviation from the expected critical frequency depends on the dimensions of the saddle. Additionally, in order to achieve a more accurate comparison of experimental and theoretical trajectories, it would be useful to develop a method of experimentally quantifying the value of the coefficient of sliding friction.

5 Conclusions

Wolfgang Paul and all other scientists interested in trapping ions may consider themselves fortunate that the RF-electric-quadrupole ion trap is not subject to the same stability limitations as its mechanical analogue. Nevertheless, I was able to demonstrate short-term stability for trapping a ball within a spinning saddle, as well as the existence of a critical frequency below which motion is decidedly unstable. As a result of this project, I also gained a greater understanding of the principles of ion trapping. Despite its differences from the ion trap, the rotating saddle is certainly a useful teaching tool and an interesting system in its own right.

References